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Some Further Generalizations of Ky Fan's Best Approximation Theorem

XIE PING DING*

Department of Mathematics, Sichuan National University, Chengdu, Sichuan, People's Republic of China 610066

AND

E. TARAFDAR

Department of Mathematics, University of Queensland, Brisbane, Queensland 4072, Australia

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In this paper, several best approximation theorems and coincidence theorems involving two mappings with point-values and set-values and two different topological vector spaces are proved. The results improve, unify, and generalize most of the recent known results in the literature.

1. INTRODUCTION

In 1969, Ky Fan [10] proved the following well-known result on best approximations:

THEOREM A. *Let X be a nonempty compact convex set in a locally convex Hausdorff topological vector space E. Let $f: X \to E$ be a continuous mapping. Then either f has a fixed point in X, or there exist a point $y_0 \in X$ and a continuous semi-norm p on E such that*

$$0 < p(y_0 - f(y_0)) = \min\{p(x - f(y_0)): x \in X\}.$$

Since then there have appeared many generalizations and applications of this theorem, e.g. see [4, 5, 8, 11, 13-17, 21-26, 29-31]. The setvalued analogues of the best approximation and fixed point theorem A were first

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obtained by Reich [24]. Recently, Prolla [23], Ha [13], Lin [17] and Carbone [4, 5] have extended Fan's theorem to two mappings and two different space settings under their own assumptions, while Reich [24–26], Sehgal-Singh [29, 30], Ha [14], Park [21, 22], Kong-Ding [17] and Ding-Tan [8] generalized Fan's theorem to continuous or upper semicontinuous set-valued mappings under different assumptions.

The result in Theorem A of Fan can be referred to as 'best approximation and fixed point result'. In the same way the results on two space settings in this paper can be referred to as 'best approximation and coincidence point results'. The coincidence problem was first studied in the topological setting in 1946 by Eilenberg and Montgomery [9] (we refer to Chang and Song [6] on the earlier development of the topic). Gorniewicz and Kucharski [12] applied their coincidence theory for k-set contraction pairs in Banach spaces to the study of fixed point theorems for setvalued mappings and boundary value problems for differential inclusions. It will also be not out of place to mention that Mawhin [19] developed the coincidence degree theory to solve the equation $Lx = Nx$, where L: $dom\, L \subset X \to Z$ is a Fredholm mapping of index zero and N: $\mathrm{dom}\, N \subset X \to Z$ is a nonlinear mapping and X and Z are Banach spaces, and applied it to boundary value problems of partial differential equations. While Tarafdar and Teo [32] developed a coincidence degree theory to solve the inclusion equation $Lx \in N(x)$, where L and N are as above.

In this paper, we obtain some best approximation theorems involving two mappings and two different spaces which improve, unify and generalize most of the known results in the literatures mentioned above.

2. Preliminaries

Let X be a nonempty set, 2^X denote the family of all subsets of X and $\mathscr{F}(X)$ denote the family of all nonempty finite subsets of X. If X is a topological space with topology T, we shall use (X, T) and $2^{(X, T)}$ to denote the sets X and 2^X respectively with emphasis on the fact that X is equipped with the topology T. If A is a subset of a topological space (X, T), we shall denote by $\mathrm{int}_{(X, T)}\,(A)$ and $\partial_{(X, T)},\,(A)$ the interior and the boundary of A in (X, T) respectively. Let (X, T) and (Y, S) be two topological spaces; a set-valued mapping $F\colon (X, T) \to 2^{(Y, S)}$ is said to be upper semicontinuous on X if for each $x_0 \in X$ and for each S-open set G in Y with $F(x) \subset G$, there exists a T-open neighborhood U of x_0 in X such that $F(x) \subset G$ for all $x \in U$. Let E be a topological vector space with topology T. We shall denote by $E^* = (E, T)^*$ the topological dual of (E, T). E^* is said to sparate points of E if for each $x \in E$ with $x \neq 0$, there exists an $f \in E^*$ such that $f(x) \neq 0$. We shall denote by $W = W(E, E^*)$ the weak topology of E and by

$\mathscr{P} = \mathscr{P}(E, T)$ the family of all continuous semi-norms on (E, T). If X is a nonempty subset of E, we shall denote by $\text{co}(X)$ the convex hull of X and by (X, T) and (X, W) the set X equipped with the relative topology of T to X and the relative topology of W to X respectively. We shall denote by $\mathbb{R}$ the set of all real numbers and if z is a complex number, we shall denote by $\text{Re}\, z$ the real part of z.

Let X be a non-empty subset of a topological vector space (E, T). For each $x \in E$, the inward set of X at x, denote by $I_X(x)$, is defined by

$$I_X(x) = \{x + r(y - x): y \in X \text{ and } r > 0\}.$$

The closure of $I_X(x)$ in (E, T), denoted by $\overline{I_X}(x)$, is called the weakly inward set of X at x.

Let (E, T) and (F, S) be two topological vector spaces. X be a nonempty convex subset of (E, T) and let $(F, S)^*$ separates points of (F, S). Following Prolla [23] (also see Mehta-Sessa [20]), a mapping $g: X \to F$ is said to be almost affine if for any $x, y \in X$ and for any $p \in \mathscr{P}(F, S)$

$$p(g(\lambda x + (1 - \lambda) y) - z) \leqslant \lambda p(g(x) - z) + (1 - \lambda)\, p(g(y) - z)$$

for all $z \in F$ and $\lambda \in [0, 1]$. Following Carbone [4], a mapping $g: X \to F$ is said to be almost quasi-convex if for each $z \in F$, each $p \in \mathscr{P}(F, S)$ and each $r > 0$, the set $\{x \in X: p(g(x) - z) < r\}$ is convex.

Obviously, each affine mapping (i.e. $g(\lambda x + (1 - \lambda) y) = \lambda g(x) + (1 - \lambda) g(y)$ for all $x, y \in X$ and $\lambda \in [0, 1]$) is almost affine and each almost affine mapping is almost quasi-convex, but not conversely.

PROPOSITION 2.1. *Let (E, T) and (F, S) be two topological vector spaces, $(F, S)^*$ separates points of (F, S) and X be a nonempty convex subset of (E, T). Let $g: X \to F$ satisfy the following condition (see Ha [13] and Lin [18]): $g^{-1}([u, v])$ is convex for any $u, v \in g(X)$, where $[u, v] = \{\lambda u + (1 - \lambda) v: \lambda \in [0, 1]\}$. Then g is almost quasi-convex.*

Proof. For any $z \in F$, $p \in \mathscr{P}(F, S)$ and $r > 0$. Let $x, y \in \{x \in X: p(g(x) - z) < r\}$. Then $p(g(x) - z)) < r$ and $p(g(y) - z) < r$. Let $u = g(x)$ and $v = g(y)$, then $x, y \in g^{-1}([u, v])$. By the assumption $g^{-1}([u, v])$ is convex. We have $[x, y] \subset g^{-1}([u, v])$. Thus for each $\lambda \in [0, 1]$ there exists $k \in [0, 1]$ such that $g(\lambda x + (1 - \lambda) y) = ku + (1 - k) v = kg(x) + (1 - k) g(y)$. It follows that

$$\begin{aligned} p(g(\lambda x + (1 - \lambda) y) - z) &= p(kg(x) + (1 - k)\, g(y) - z) \\ &\leqslant kp(g(x) - z) + (1 - k)\, p(g(y) - z) \\ &< kr + (1 - k) r = r. \end{aligned}$$

Hence the set $\{x \in X: p(g(x)-z)<r\}$ is convex and g is almost quasi-convex. ∎

In order to prove our main theorems, the following Lemmas are needed.

LEMMA 2.1 *Let (E, T) and (F, S) be Hausdorff topological vector spaces, $(E, T)^*$ and $(F, S)^*$ the topological duals of (E, T) and (F, S) respectively such that $(F, S)^*$ separates points of F. Let X be a nonempty subset of (E, T), $W(E, E^*)$ and $W(F, F^*)$ be the weak topology on E and F respectively. Suppose that g: $(X, W(E, E^*)) \to (F, W(F, F^*))$ is continuous $G: (X, W(E, E^*)) \to 2^{(F,S)}$ is u.s.c. such that each $G(x)$ is S-compact and $p \in \mathscr{P}(F, S)$. Then the function $V: (X, W(E, E^*)) \to \mathbb{R}$ defined by*

$$V(x)=d_p(g(x), G(x))=\inf\{p(g(x)-z): z \in G(x)\}$$

is lower semicontinuous (in short, l.s.c.), i.e. $V: X \to \mathbb{R}$ is weakly l.s.c.

Proof. Define a function $h: (X, W(E, E^*)) \times (F, S) \to \mathbb{R}$ by

$$h(x, z)=p(g(x)-z), \qquad \text{for} \quad (x, z) \in X \times F.$$

For each $r \in \mathbb{R}$, let $A(r)=\{(x, z) \in X \times F: h(x, z) \leqslant r\}$. Let $\{(x_\alpha, z_\alpha)\}_{\alpha \in \Delta}$ be a net in $A(r)$ and $(x, z) \in X \times F$ such that $x_\alpha \to x$ in $W(E, E^*)$-topology and $z_\alpha \to z$ in S-topology. By the continuity of $g, g(x_\alpha) \to g(x)$ in $W(F, F^*)$-topology. By the Corollary of Hahn-Banach theorem (e.g. see [27, Corollary 2, p. 29]), there exists $f^* \in (F, S)^*$ such that $f^*(g(x)-z)=p(g(x)-z)$ and $|f^*(z)| \leqslant p(z)$ for all $z \in F$.

Since $g(x_\alpha)-z_\alpha \to g(x)-z$ in $W(F, F^*)$-topology, we have

$$\begin{aligned} h(x, z) &= p(g(x)-z)=f^*(g(x)-z) \\ &= \operatorname{Re} f^*(g(x)-z) \\ &= \lim_\alpha \operatorname{Re} f^*(g(x_\alpha)-z_\alpha) \\ &\leqslant \liminf_\alpha |f^*(g(x_\alpha)-z_\alpha)| \\ &\leqslant \liminf_\alpha p(g(x_\alpha)-z_\alpha) \\ &= \liminf_\alpha h(x_\alpha, z_\alpha) \leqslant r \end{aligned}$$

So that $(x, z) \in A(r)$. Thus $A(r)$ is closed in $(X, W(E, E^*)) \times (F, S)$ and h is l.s.c. on $(X, W(E, E^*)) \times (F, S)$. Hence by Theorem 2.5.1 of Aubin [1. p. 67] the function V is l.s.c. on $(X, W(E, E^*))$. ∎

Remark. 2.1. When $(E, T)=(F, S)$ and $g=I$, the identity mapping, Lemma 2.1 reduces to Lemma 4 of Ding-Tan [8].

LEMMA 2.2. *Let (E, T) and (F, S) be Hausdorff topological vector spaces and $(F, S)^*$ separates points of (F, S). Let X be a nonempty $W(E, E^*)$-compact subset of E, $g: (X, W(E, E^*)) \to (F, W(F, F^*))$ continuous and $G: (W(E, E^*)) \to 2^{(F, S)}$ u.s.c. such that for each $x \in X$, $G(x)$ is S-compact and convex. If for each $p \in \mathscr{P}$ (F, S), there exists $x_p \in X$ such that $d_p(g(x_p), G(x_p))=0$, then g and G have a coincidence point in X, i.e. there exists $x_0 \in X$ such that $g(x_0) \in G(x_0)$.*

Proof. By the assumption and Lemma 2.1, for each $p \in \mathscr{P}(F, S)$ the set $A(p)=\{x \in X: d_p(g(x), G(x))=0\}$ is nonempty and $W(E, E^*)$-closed. If $\{p_1, ..., p_n\}$ is a finite subset of $\mathscr{P}(F, S)$, then $\sum_{i=1}^n p_i \in \mathscr{P}(F, S)$ and $A(\sum_{i=1}^n p_i) \subset \bigcap_{i=1}^n A(p_i)$. Thus the family $\{A(p): p \in \mathscr{P}(F, S)\}$ has the finite intersection property. By the $W(E, E^*)$-compactness of X, $\bigcap_{p \in \mathscr{P}(F, S)} A(p) \neq \phi$. Take any $\hat{x} \in \bigcap_{p \in \mathscr{P}(F, S)} A(p)$ then $d_p(g(\hat{x}), G(\hat{x}))=0$ for all $p \in \mathscr{P}(F, S)$. Since $G(\hat{x})$ is S-compact, by Lemma 5 of Ding-Tan [8], $g(\hat{x}), \in G(\hat{x})$. ∎

The following general minimax inequality is Theorem 1 of Ding-Tan [7].

LEMMA 2.3. *Let X be a non-empty convex subset of a topological vector space and $\varphi: X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(i) *for each $x \in X$, $y \to \varphi(x, y)$ is l.s.c. on each compact subset C of X,*

(ii) *for each $A \in \mathscr{F}(X)$ and for each $y \in co(A)$, $\min_{x \in A} \varphi(x, y) \leqslant 0$,*

(iii) *there exist a nonempty compact convex subset X_0 of X and a non-empty compact subset K of X such that for each $y \in X \backslash K$, there is an $x \in co(X_0 \cup \{y\})$ with $\varphi(x, y)>0$.*

Then there exists $\hat{y} \in K$ such that $\varphi(x, \hat{y}) \leqslant 0$ for all $x \in X$.

The following result is Theorem 1 of Ha [14].

LEMMA 2.4. *Let E, F be Hausdorff topological vector spaces, $X \subset E$, $Y \subset F$ be nonempty convex sets and Y be compact. Let $G: X \to 2^Y$ be u.s.c. with nonempty closed and convex values and $\varphi: X \times Y \to \mathbb{R}$ be such that*

(a) *for each $x \in X$, $y \to \varphi(x, y)$ is l.s.c. in Y*

(b) *for each $y \in Y$, $x \to \varphi(x, y)$ is quasi-concave in X.*

Then

$$\inf_{y \in Y} \sup_{x \in X} \varphi(x, y) \leqslant \sup_{u \in G(x), x \in X} \varphi(x, u).$$

3. Approximation Theorems and Coincidence Theorems

In this section, we shall prove several approximation theorems and coincidence theorems for two mappings and two different space settings.

Theorem 3.1. *Let (E, T) and (F, S) be Hausdorff topological vector spaces, X be a nonempty convex subset of E and $(F, S)^*$ separate points of (F, S) Let $g:(X, W(E, E^*)) \to (F, W(F, F^*))$ be almost quasi-convex and continuous and $f: (X, W(E, E^*)) \to (F, S)$ be continuous where $W(E, E^*)$ and $W(F, F^*)$ denote the weak topology on E and F respectively. Suppose that there exist a nonempty $W(E, E^*)$-compact convex subset X_0 of X and a nonempty $W(E, E^*)$-compact subset K of X such that for each $y \in X \backslash K$ and for each $p \in \mathscr{P}(F, S)$ there is an $x \in co(X_0 \cup \{y\})$ such that $p(g(x) - f(y)) < p(g(y) - f(y))$. Then either*

(a) *there exists $x_0 \in K$ such that $g(x_0) = f(x_0)$ or*

(b) *there exist $p \in \mathscr{P}(F, S)$ and $x^* \in K$ such that $g(x^*) \in \partial g(X)$ and $0 < p(g(x^*) - f(x^*)) = \min\{p(g(x) - f(x^*)): x \in X\}$.*

If, in addition, we further assume that $g(X)$ is convex, then we have

$$p(g(x^*) - f(x^*)) = \min\{p(z - f(x^*)): z \in \overline{I_{g(x)}}(g(x^*))\}.$$

Proof. For $p \in \mathscr{P}(F, S)$, define a function $\varphi:(X, W(E, E^*)) \times (X, W(E, E^*)) \to \mathbb{R}$ by $\varphi(x, y) = p(g(y) - f(y)) - p(g(x) - f(y))$. Then it is easy to see that for each $x \in X$, $y \to \varphi(x, y)$ is weakly l.s.c. For each $A \in \mathscr{F}(X)$ and $y \in co(A)$, we must have $\min_{x \in A} \varphi(x, y) \leqslant 0$; if this were not true, then there would exist $A = \{x_1, ..., x_n\} \in \mathscr{F}(X)$ and $y = \sum_{i=1}^n \lambda_i x_i \in co(A)$ with $\lambda_1, ..., \lambda_n \geqslant 0$ and $\sum_{i=1}^n \lambda_i = 1$ such that $\varphi(x_i, y) = p(g(y) - f(y)) - p(g(x_i) - f(y)) > 0$ for all $i = 1, ..., n$, that is, $A \subset \{x \in X: p(g(x) - f(y)) < p(g(y) - f(y))\}$. Since g is almost quasi-convex, we have

$$y \in co(A) \subset \{x \in X: p(g(x) - f(y)) < p(g(y) - f(y))\}$$

which is impossible. By assumption, there exist a nonempty $W(E, E^*)$-compact convex subset X_0 of X and a nonempty $W(E, E^*)$-compact subset K of X such that for each $y \in X \backslash K$, there is an $x \in co(X_0 \cup \{y\})$ with $\varphi(x, y) > 0$. Hence, by Lemma 2.3, there exists $x_p \in K$ such that $\varphi(x, x_p) \leqslant 0$ for all $x \in X$, i.e.,

$$p(g(x_p) - f(x_p)) \leqslant p(g(x) - f(x_p)) \qquad \text{for all} \quad x \in X.$$

If for each $p \in \mathscr{P}(F, S)$, $p(g(x_p) - f(x_p)) = 0$, then as $x_p \in K$ for each $p \in \mathscr{P}(F, S)$, it follows from Lemma 2.2 with $G = \{f\}$ being a single valued mapping that there exists an $x_0 \in K$ such that $g(x_0) = f(x_0)$.

Otherwise there exists $p \in \mathscr{P}(F, S)$ and, $x_p = x^* \in K$ such that

$$0 < p(g(x^*) - f(x^*)) = \min\{p(g(x) - f(x^*)) : x \in X\}$$
$$= \min\{p(z - f(x^*)) : z \in g(X)\}.$$

and hence $f(x^*) \notin g(X)$. If $g(x^*) \in int_F(g(X))$, then there exist λ, $0 < \lambda < 1$, such that $z = \lambda g(x^*) + (1-\lambda) f(x^*) \in g(X)$. It follows that

$$\begin{aligned} 0 &< p(g(x^*) - f(x^*)) \leqslant p(z - f(x^*)) \\ &= p(\lambda g(x^*) + (1-\lambda) f(x^*) - f(x^*)) \\ &= \lambda p(g(x^*) - f(x^*)) < p(g(x^*) - f(x^*)) \end{aligned}$$

which is impossible and so $g(x^*) \in \partial g(X)$.

Now assume that $g(X)$ is convex. Let $z \in I_{g(X)}(g(x^*)) \backslash g(X)$. Since $g(X)$ is convex, there exist $y \in X$ and $r > 1$ such that $z = g(x^*) + r(g(y) - g(x^*))$. Suppose that

$$p(z - f(x^*)) < p(g(x^*) - f(x^*)).$$

Then we have $g(y) = (1 - 1/r) g(x^*) + 1/r z \in g(X)$ and

$$\begin{aligned} p(g(x^*) - f(x^*)) &\leqslant p(g(y) - f(x^*)) \\ &\leqslant p\left(\left(1 - \frac{1}{r}\right) g(x^*) + \frac{1}{r} z - f(x^*)\right) \\ &\leqslant \left(1 - \frac{1}{r}\right) p(g(x^*) - f(x^*)) + \frac{1}{r} p(z - f(x^*)) \\ &< p(g(x^*) - f(x^*)) \end{aligned}$$

which is impossible and hence

$$p(g(x^*) - f(x^*)) \leqslant p(z - f(x^*)) \qquad \text{for all} \quad z \in I_{g(X)}(g(x^*)).$$

Since p is continuous, we must have

$$p(g(x^*) - f(x^*)) = \min\{p(z - f(x^*)) : z \in \overline{I_{g(X)}}(g(x^*)). \quad \blacksquare$$

Remark. 3.1. Theorem 3.1 improves and generalizes Theorem 1 and Theorem 2 of Lin [18], Theorem 3 of Ha [13] with the condition being replaced by (c); Theorem of Prolla [23], Theorems 1 and 3 of Sehgal-Singh-Smithson [31]. Theorems 1 and 2 of Carbone [4], Theorem 2.1 of Carbone [5] and Theorem 2 of Kapoor [15] in several aspects.

THEOREM 3.2. *Let (E, T) and (F, S) be Hausdorff topological vector spaces, X be a nonempty convex subset of E and $(F, S)^*$ separate points of (F, S). Let $g: (X, W(E, E^*)) \to (F, W(F, F^*))$ be almost quasi-convex and continuous and $f: (X, W(E, E^*)) \to (F, S)$ be continuous. Suppose that*

(1) *there exist a nonempty $W(E, E^*)$-compact convex subset X_0 of X and a nonempty $W(E, E^*)$-compact subset K of X such that for each $y \in X \backslash K$ and for each $p \in \mathscr{P}(F, S)$, there is an $x \in co(X_0 \cup \{y\})$ with $p(g(x) - f(y)) < p(g(y) - f(y))$,*

(2) *for each $x \in X$ there exists a number λ (real or complex, depending on whether the vector space F is real or complex) such that $|\lambda| < 1$ and $\lambda g(x) + (1 - \lambda) f(x) \in g(X)$.*

Then there exists a point $x_0 \in K$ such that $g(x_0) = f(x_0)$.

Proof. Suppose that $g(x) \neq f(x)$ for all $x \in K$. By Theorem 3.1, there exists $p \in \mathscr{P}(F, S)$ and $x^* \in K$ such that

$$0 < p(g(x^*) - f(x^*)) \leqslant p(z - f(x^*) \qquad \text{for all} \quad z \in g(X).$$

Now by the hypotheses, there exists a number λ such that $|\lambda| < 1$ and $z = \lambda g(x^*) + (1 - \lambda) f(x^*) \in g(X)$ and hence

$$\begin{aligned} p(g(x^*) - f(x^*)) &\leqslant p(\lambda g(x^*) + (1 - \lambda) f(x^*) - f(x^*)) \\ &\leqslant |\lambda| \, p(g(x^*) - f(x^*)) \end{aligned}$$

which contradicts the fact $|\lambda| < 1$ and $p(g(x^*) - f(x^*)) > 0$. ∎

Remark 3.2. Theorem 3.2 generalizes Theorems 3 and 4 of Lin [18] and Theorem 4 of Ha [13].

THEOREM 3.3. *Let X be a nonemtpy $W(E, E^*)$-compact convex subset of a Hausdorff topological vector space (E, T) and (F, S) be a Hausdorff locally convex topological vector space. Let $g: (X, W(E, E^*)) \to (F, W(F, F^*))$ be almost quasi-convex and continuous and $G: (X, W(E, E^*)) \to 2^{(F, S)}$ be u.s.c. such that each $G(x)$ is S-compact and convex. Then either (a) there exist $\hat{x} \in X$ such that $g(\hat{x}) \in G(\hat{x})$; or (b) there exist $x_0 \in X$, $u_0 \in G(x_0)$ and $p \in \mathscr{P}(F, S)$ such that*

$$0 < p(g(x_0) - u_0) \leqslant p(g(x) - u_0) \qquad \textit{for all} \quad x \in X.$$

In addition, if we assume that $g(X)$ is convex, then we have

$$0 < p(g(x_0) - u_0) \leqslant p(z - u_0) \qquad \textit{for all} \quad z \in \overline{I_{g(X)}}(g(x_0)).$$

Proof. Suppose $g(x) \notin G(x)$ for all $x \in X$, then $0 \notin g(x) - G(x)$ and $g(x) - G(x)$ is a S-compact convex subset of F for each $x \in X$. By Theorem 3.4 of Rudin [28, p.58], there exist $\delta_x > 0$ and a continuous linear functional $p_x \in F^*$ such that

$$d_{p_x}(g(x), G(x)) = \inf_{u \in G(x)} |p_x(g(x) - u)| > \delta_x.$$

By Lemma 2.1, there exists an open $W(E, E^*)$-neighbourhood $N(x)$ of x such that

$$d_{p_x}(g(z), G(z)) > \delta_x \qquad \text{for all} \quad z \in N(x).$$

Since $X = \bigcup_{x \in X} N(x)$ and X is $W(E, E^*)$-compact, there exists a finite set $\{x_1, ..., x_n\} \subset X$ such that $X \subset \bigcup_{i=1}^{n} N(x_i)$. Let $p = \max\{|p_{x_i}| : i = 1, ..., n\}$ and $\delta = \min\{\delta_{x_i} : i = 1, ..., n\}$, then $p \in \mathscr{P}(F, S)$. For each $x \in X$, there exists $j \in \{1, ..., n\}$ such that $x \in N(x_j)$; it follows that for each $u \in G(x)$

$$p(g(x) - u) = \max_{1 \leqslant i \leqslant n} |p_{x_i}(g(x) - u)| \geqslant |p_{x_j}(g(x) - u)| \geqslant d_{p_{x_j}}(g(x), G(x))$$

so that

$$d_p(g(x), G(x)) \geqslant d_{p_{x_j}}(g(x), G(x)) > \delta_{x_j} \geqslant \delta.$$

Hence we have

$$d_p(g(x), G(x)) > \delta \qquad \text{for all} \quad x \in X.$$

Now define a function φ: $(X, W(E, E^*)) \times (F, S) \to \mathbb{R}$ by

$$\varphi(x, z) = \min_{y \in X} p(g(y) - z) - p(g(x) - z).$$

By the continuity of g and p, it is easy to see that φ is continuous. Thus the condition (a) of Lemma 2.4 is satisfied. For each $z \in (F, S)$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} B &= \{x \in X : \varphi(x, z) > \lambda\} \\ &= \{x \in X : p(g(x) - z) < \min_{y \in X} p(g(y) - z) - \lambda\}. \end{aligned}$$

Since g is almost quasi-convex, therefore B is convex and so the condition (b) of Lemma 2.4 is satisfied. By proposition 3.1.11 of Aubin-Ekeland [2. p.112], $G(X) = \bigcup_{x \in X} G(x)$ is S-compact. As the assumptions on X, G and the graph $G_r(G)$ of G remain unchanged in the completion of F, without loss of generality, we may assume that (F, S) is complete. Let $Y = \overline{\text{co}}(G(X))$, then $Y \subset F$ is S-compact and convex. By applying Lemma 2.4, we have

$$\inf_{z \in Y} \sup_{x \in X} \varphi(x, z) \leqslant \sup_{u \in G(x),\, x \in X} \varphi(x, u)$$

Since for each $z \in Y$,

$$\sup_{x \in X} \varphi(x, z) = \sup_{x \in X} \left[\min_{y \in X} p(g(y) - z) - p(g(x) - z)\right]$$
$$= \min_{g \in X} p(g(y) - z) - \min_{x \in X} p(g(x) - z) = 0,$$

We have

$$\sup_{u \in G(x),\, x \in X} \varphi(x, u) \geqslant 0$$

Since φ: $(X, W(E, E^*)) \times (F, S) \to \mathbb{R}$ is continuous and the graph $Gr(G)$ is compact in $(X, W(E, E^*)) \times (F, S)$, there exists $(x_0, u_0) \in Gr(G)$ such that

$$\varphi(x_0, u_0) = \min_{y \in X} p(g(y) - u_0) - p(g(x_0) - u_0) \geqslant 0.$$

It follows that $u_0 \in G(x_0)$ and

$$\delta < d_p(g(x_0), G(x_0)) \leqslant p(g(x_0) - u_0) \leqslant p(g(x) - u_0) \qquad \text{for all} \quad x \in X.$$

Now further assume that $g(X)$ is convex. For $x \in I_{g(X)}(g(x_0)) \backslash g(X)$ there exist $v \in g(X)$ and $r > 1$ such that $z = g(x_0) + r(v - g(x_0))$. Suppose that $p(z - u_0) < p(g(x_0) - u_0)$. Since $v = 1/rz + (1 - 1/r)\, g(x_0) \in g(X)$, we must have

$$\begin{aligned} p(g(x_0) - u_0) &\leqslant p(v - u_0) = p\left(\frac{1}{r} z + \left(1 - \frac{1}{r}\right) g(x_0) - u_0\right) \\ &\leqslant \frac{1}{r} p(z - u_0) + \left(1 - \frac{1}{r}\right) p(g(x_0) - u_0) \\ &< p(g(x_0) - u_0) \end{aligned}$$

which is impossible. Therefore we must have

$$p(g(x_0) - u_0) \leqslant p(z - u_0) \qquad \text{for all} \quad z \in I_{g(X)}(g(x_0)).$$

Since p is continuous, we have

$$0 < p(g(x_0) - u_0) \leqslant p(z - u_0) \qquad \text{for all} \quad z \in \overline{I_{g(X)}}(g(x_0)). \quad \blacksquare$$

Remark. 3.3. If one take $E = F$ and $g: (X, W(E, E^*)) \to E$ as the identity mapping, Theorem 3.3 reduces to Theorem 4 of Ding-Tan [7]. Theorem 3.3 also generalizes Theorem 3 of Ha [14], Theorem 3 of Park [22], Corollary 1 of Browder [3], Theorem 3.1 of Reich [25] and Theorem 1 of Fan [10] to two mappings and two different space setting. Komiya [16] also has obtained some similar results on two setvalued mappings, we note that his results are not comparable with ours.

As an equivalent version of Theorem 3.3, we have

THEOREM 3.4. *Let X be a nonempty $W(E, E^*)$-compact convex subset of a Hausdorff topological vector space (E, T) and (F, S) be a Hausdorff locally convex topological vector space. Suppose that $g: (X, W(E, E^*)) \to (F, W(F, F^*))$ almost quasi-convex and continuous and $G: (X, W(E, E^*)) \to 2^{(F,S)}$ u.s.c. such that each $G(x)$ is S-compact and convex. If one of the following two conditions is satisfied:*

(1) *for each $p \in \mathscr{P}(F, S)$, each $x \in X$ with $d_p(g(x), G(x)) > 0$ and each $u \in G(x)$,*

$$d_p(u, g(X)) < p(g(x) - u);$$

or

(2) *$g(X)$ is convex and for each $p \in \mathscr{P}(F, S)$, each $x \in X$ with $d_p(g(x), G(x)) > 0$ and each $u \in G(x)$,*

$$d_p(u, \overline{I_g(X)}(g(x))) < p(g(x) - u);$$

then there exists a point $\hat{x} \in X$ such that $g(\hat{x}) \in G(\hat{x})$.

Remark 3.4. Theorem 3.4 generalizes Theorem 5 of Ding-Tan [8] and Theorem 2 of Reich [26] to two mappings and two different space settings.

THEOREM 3.5. *Let X be a nonempty $W(E, E^*)$-compact convex subset of a Hausdorff topological vector space (E, T) and (F, S) be a Hausdorff locally convex topological vector space. Let $g: (X, W(E, E^*)) \to (F, W(F, F^*))$ be almost quasi-convex and continuous such that $g(X)$ is convex and $G: (X, W(E, E^*)) \to 2^{(F,S)}$ be u.s.c. such that each $G(x)$ is S-compact convex. Further, assume that the following condition is satiesfied:*

(a) *for each $x \in X$ with $g(x) \in \partial_{(F,W)} g(X) \setminus G(x)$ and $u \in G(x)$, there exists a number λ (real or complex, depending on whether the vector space F is real or complex) with $|\lambda| < 1$ such that $\lambda g(x) + (1 - \lambda) u \in \overline{I_{g(X)}}(g(x))$.*

Then there exists $\hat{x} \in X$ such that $g(\hat{x}) \in G(\hat{x})$.

Proof. Suppose $g(x) \notin G(x)$ for all $x \in X$. By Theorem 3.3, there exist $x_0 \in X$, $u_0 \in G(x_0)$ and $p \in \mathscr{P}(F, S)$ such that

$$0 < p(g(x_0) - u_0) \leqslant p(z - u_0) \qquad \text{for all} \quad z \in \overline{I_{g(X)}}(g(x_0)).$$

Case 1. If $g(x_0) \in int_{(F, W)}(g(X))$, then $I_{g(X)}(g(x_0)) = F$ and hence $u = \frac{1}{2}g(x_0) + \frac{1}{2}u_0 \in F = I_{g(X)}(g(x_0))$, so that

$$0 < p(g(x_0) - u_0) \leqslant p(u - u_0) = \tfrac{1}{2}p(g(x_0) - u_0)$$

which is absurd.

Case 2. If $g(x_0) \in \partial_{(F, W)} g(X)$, $g(x_0) \in \partial_{(F, W)} g(X) \backslash G(x_0)$ so that by condition (a), there exists λ with $|\lambda| < 1$ such that $\lambda g(x_0) + (1 - \lambda) u_0 \in \overline{I_{g(X)}}(g(x_0))$. It follows that

$$\begin{aligned} 0 < p(g(x_0) - u_0) &\leqslant p(\lambda g(x_0) + (1 - \lambda) u_0 - u_0) \\ &\leqslant |\lambda| \, p(g(x_0) - u_0) \end{aligned}$$

which is again a contradiction. Therefore there exists $\hat{x} \in X$ such that $g(\hat{x}) \in G(\hat{x})$. ∎

Remark 3.5. If one takes $E = F =$ a locally convex Hausdorff topological vector space and g as the identity mapping, Theorem 3.5 reduces to Theorem 6 of Ding-Tan [8] and hence Theorem 3.1 improves and generalizes Theorem 4 of Ha [14], Theorem 3.1 of Reich [25], Theorem 4 of Park [22] and Theorem 3 of Fan [10] to two mappings and two different space settings.

If the assumptions "X is $W(E, E^*)$-compact, $g: (X, W(E, E^*)) \to (F, W(F, F^*))$ is continuous and $G: (X, W(E, E^*)) \to 2^{(F, S)}$ are u.s.c." is replace by the assumption "X is T-compact, $g: (X, T) \to (F, S)$ is continuous and $G: (X, T) \to 2^{(F, S)}$ are u.s.c." in Theorems 3.3, 3.4 and 3.5, then, by the same argument, we obtain the following theorems.

THEOREM 3.3'. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E and F be a Hausdorff locally convex topological vector space. Let $g: X \to 2^F$ be almost quasi-convex and continuous and $G: X \to 2^F$ be u.s.c. such that each $G(x)$ is compact and convex. Then either (a) there exist $\hat{x} \in X$ such that $g(\hat{x}) \in G(\hat{x})$; or (b) there exist $x_0 \in X$, $u_0 \in G(x_0)$ and $p \in \mathscr{P}(F)$ such that*

$$0 < p(g(x_0) - u_0) \leqslant p(g(x) - u_0) \qquad \textit{for all} \quad x \in X.$$

In addition if $g(X)$ *is convex, then we have*

$$0 < p(g(x_0) - u_0) \leqslant p(z - u_0) \qquad \text{for all} \quad z \in \overline{I_{g(X)}}(g(x_0)).$$

THEOREM 3.4'. *Let* X *be a nonemtpy compact convex subset of a Hausdorff topological vector space* E *and* F *be a Hausdorff locally convex topological space. Suppose that* $g: X \to F$ *is almost quasi-convex and continuous and* $G: X \to 2^F$ *be u.s.c. such that each* $G(x)$ *is compact and convex. If one of the following two conditions is satisfied*:

(1) *for each* $p \in \mathscr{P}(F)$, *each* $x \in X$ *with* $d_p(g(x), G(x)) > 0$ *and each* $u \in G(x)$

$$d_p(u, g(X)) < p(g(x) - u);$$

(2) $g(X)$ *is convex and for each* $p \in \mathscr{P}(F)$, *each* $x \in X$ *with* $d_p(g(x), G(x)) > 0$ *and each* $u \in G(x)$, $d(u, \overline{I_{g(X)}}(g(x))) < p(g(x) - u)$; *then there exists a point* $\hat{x} \in X$ *such that* $g(\hat{x}) \in G(\hat{x})$.

THEOREM 3.5'. *Let* X *be a nonempty compact convex subset of a Hausdorff topological vector space and* F *be a Hausdorff locally convex topological vector space. Let* $g: X \to F$ *be almost quasi-convex and continuous such that* $g(X)$ *is convex and* $G: X \to 2^F$ *be u.s.c. such that each* $G(x)$ *is compact and convex. Suppose the following condition is satisfied*:

(a) *for each* $x \in X$ *with* $g(x) \in \partial_F(g(X)) \backslash G(x)$ *and* $u \in G(x)$, *there exists a number* λ (*real or complex, depending on whether the vector space* F *is real or complex*) *with* $|\lambda| < 1$ *such that* $\lambda g(x) + (1 - \lambda) u \in \overline{I_{g(X)}}(g(x))$. *Then there exists* $\hat{x} \in X$ *such that* $g(\hat{x}) \in G(\hat{x})$.

COROLLARY 3.1 *Let* X *be a nonempty compact convex subset of a Hausdorff topological vector space* E *and* F *be a Hausdorff locally convex topological vector space. Let* $g: X \to F$ *be almost quasi-convex and continuous and* $G: X \to 2^F$ *be u.s.c. with nonempty closed convex values such that*

(a) *for each* $x \in X$, $G(x) \cap g(X) \neq \phi$,

(b) $g(X)$ *is convex*.

Then there exists a point $\hat{x} \in X$ *such that* $g(\hat{x}) \in G(\hat{x})$.

Proof. Define a mapping $H: X \to 2^F$ by

$$H(x) = G(x) \cap g(X) \qquad \text{for each} \quad x \in X.$$

Then for each $x \in X$, $H(x)$ is non-empty compact convex. Since G is u.s.c. and $g(X)$ is compact, H is also u.s.c. Note that for each $p \in \mathscr{P}(F)$ and each

$x \in X$ with $d_p(g(x), H(x)) > 0$ and each $u \in H(x) = G(x) \cap g(X)$, we have $d_p(u, g(X)) = 0 < d_p(g(x), H(x)) \leqslant p(g(x) - u)$. By Theorem 3.4', there exists a point $\hat{x} \in X$ such that $g(\hat{x}) \in H(\hat{x}) = G(\hat{x}) \cap g(X) \supset G(\hat{x})$. ∎

Remark 3.7. Corollary 3.1 improves Theorem 2.2 of Mehta-Sessa [20].

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